The Homotopy Hypothesis

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May 14, 2025

Following [Yan].

"Infinity groupoids are homotopy types."

From a topological space we form the singular simplicial set whose n simplicies are

$$\text{Sing}(X)_n \quad := \quad \text{Hom}_{\text{Top}}\left(\Delta^n, X\right)$$

this is a functor

$$\text{Sing} : \text{Top} \to \text{sSet}$$

when taking simplicial homology we take the $\mathbb{Z}$ span on the n-simplicies. The image of this functor lands in ∞-groupoids [Yan, Prop 2.2.7]. Note that this is not a fully faithful functor, think of a homology sphere.

This functor has a left adjoint

$$| - | : \text{sSet} \to \text{CW} \to \text{Top}$$

called geometric realisation which factors through the category of CW complexes. This functor is also not fully faithful. This functor decends to a fully faithful functor *when restricted to* ∞-groupoids and taking the homotopy category [Yan, Thm 2.2.11].

$$hGrp_\infty \to hCW \to h\text{Top}$$

and thus gives an equivalence of categories on its essential image, with inverse given by Sing. Thus (homotopy types of) ∞-groupoids are *just* homotopy types of CW complexes. I think its essential image is CW complexes right? The problem is we have groupoids not just simplicial sets. Do we get everything. If we do we can conclude that homotopy theory (study of homotopy types of CW complexes) is just the theory of infinity groupoids. By Whiteheads theorem homotopy types of CW complexes are "just" *weak* homotopy types of topological spaces.

If we take the suitable infinity category structure on the two categories we actually get an equivalence of infinity categories [Yan, Thm 3.3.3]

$$\text{Grpd}_\infty \cong \text{CW}_\infty$$

References

[Yan] Lior Yanovski. A Path to Infinity – An Introduction to -Categories.